

RESEARCH ARTICLE

## Fast wavelet transforms of Haar's wavelet

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### Abstract

Wavelets have many good properties such as time-frequency localization and compact support and so on. This study focuses on the Haar's wavelet. We will show that the analysis and synthesis of Haar's wavelet with its signal by MATLAB programming. The base of this study is followed from multiresolution analysis.

**Keywords:** Approximation, Haar's wavelet, MATLAB programming, multiresolution analysis.

### Introduction

Wavelets are a relatively recent development in applied mathematics and their name itself was coined approximately a decade ago (Morlet *et al.*, 1982, 1983; Grossmann and Morlet, 1984; Mallat, 1989a). In recent years interest in them has grown at an explosive rate. There are several reasons for their present success. On the one hand, the concept of wavelets can be viewed as a synthesis of ideas which originated during the last twenty or thirty years in engineering (subband coding), physics (coherent states, renormalization group) and pure mathematics (study of Calderon-Zygmund operators). As a consequence of these interdisciplinary origins, wavelets appeal to scientists and engineers of many different backgrounds. On the other hand, wavelets are a fairly simple mathematical tool with a great variety of possible applications. Already they have led to exciting applications in signal analysis (sound, images) (Kronland-Martinet *et al.*, 1987; Morlet and Grossmann, 1987, Mallat, 1989a, b) and numerical analysis (fast algorithms for integral transforms by Beylkin *et al.*, 1991); many other applications are being studied. This wide applicability also contributes to the interest they generate. Wood (2003) developed the wavelet in the Hilbert C\*-module case. Also Packer and Rieffel (2004) and Liu *et al.* (2012) studied this concept in  $L^2(\mathbb{R}^d)$ . Keeping the above facts in view, this study was aimed with the following objectives:

1. The algorithm of approximation and detail of analysis in MATLAB.
2. The algorithm of approximation and detail of synthesis in MATLAB.

### Materials and methods

This study is mainly based on multiresolution analysis. For more details about following concepts and proof of Theorems one can see Daubechies (1992), Mallat (1999) and Wojtaszczyk (1997). In the sequel, we denoted the integer, real and complex numbers by  $Z$ ,  $R$  and  $C$  respectively.

Definition 1. Let  $H$  be a Hilbert space

i) A family  $(x_j)_{j \in J} \subset H$  is an orthonormal system if

$$\forall i, j \in J : \langle x_i, x_j \rangle = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

ii) A family  $(x_j)_{j \in J} \subset H$  is total if for all  $x \in H$  the following implication holds

$$(\forall j \in J : \langle x, x_j \rangle = 0) \Rightarrow x = 0.$$

iii) A total orthonormal system is called orthonormal basis.

Definition 2. Let  $H$  and  $H_0$  be Hilbert spaces and  $T: H \rightarrow H_0$  be a bounded linear operator. Then  $T$  is an (orthogonal) projection if  $T = T^* = T^2$ .

Definition 3. Let  $(X, \beta, \mu)$  be a measure space.

i)  $f: X \rightarrow \mathbb{R}^n$  is called measurable if  $f^{-1}(B) \in \beta$  for all Borel set  $B$ . The space of measurable functions  $X \rightarrow C$  is denoted by  $M(X)$ .

ii) For  $1 \leq p < \infty$ , the space  $L^p(X, \mu) = L^p(X)$  is defined by

$$L^p(X) = \{f \in M(X) : \int_X |f(x)|^p d\mu(x) < \infty\}$$

endowed with the  $L^p$ -norm

$$\|f\|_p = \left( \int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}$$

iii) For  $p = \infty$  we let

$$L^\infty(X) = \{f \in M(X) : \|f\|_\infty < \infty\}$$

Where

$$\|f\|_\infty = \inf\{\alpha : |f(x)| < \alpha \text{ a. e.}\}$$

$L^p(X)$  is complete with respect to the metric  $d(f, g) = \|f - g\|_p$ ; this makes  $L^p(X)$  a Banach space.  $L^2(X)$  is a Hilbert space, with scalar product

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} d\mu(x).$$

For more details one can see Rudin (1987).

**Definition 4.** The Fourier transform of a function  $f \in L^1(\mathbb{R})$  is  $\hat{f}(\omega) = \int f(x)e^{-2\pi i\omega x} d\mu(x)$ , for  $\omega \in \mathbb{R}$ . If  $\hat{f} \in L^1(\mathbb{R})$  then  $f$  is continuous with  $f(x) = \int \hat{f}(\omega)e^{2\pi i\omega x} d\mu(\omega)$ .

**Definition 5.** Let  $\omega, x, a \in \mathbb{R}$ , with  $a \neq 0$ . We define  
 $(T_x f)(y) = f(y - x)$ , Translation operator  
 $(D_a f)(y) = |a|^{-1/2} f(a^{-1}y)$ , Dilation operator  
 $(M_\omega f)(y) = e^{-2\pi i\omega y} f(y)$ , Modulation operator

All these operators are easily checked to be unitary on  $L^2(\mathbb{R})$ .

Dilation and translation have a simple geometrical meaning.  $T_x$  amounts to shifting a graph of a function by  $x$ , whereas  $D_a$  is a dilation of the graph by  $a$  along the  $x$ -axis, with a dilation by  $|a|^{-1/2}$  along the  $y$ -axis: If  $\text{supp} f = [0, 1]$ , then  $\text{supp}(D_a f) = [0, a]$  and  $\text{supp}(T_x f) = [x, x+1]$ , where  $\text{supp}(g) = \{x : g(x) \neq 0\}$ .

**Definition 6.** If  $f, g$  are complex-valued functions defined on  $\mathbb{R}$ , their convolution  $f * g$  is the function  $(f * g)(x) = \int f(x - y)g(y)d\mu(y)$ , provided that the integral exists.

**Definition 7.** A multiresolution analysis (MRA) is a sequence of closed subspaces  $(V_j)_{j \in \mathbb{Z}}$  of  $L^2(\mathbb{R})$  with the following list of properties:

1.  $\forall j \in \mathbb{Z} : V_j \subset V_{j+1}$ ,
2.  $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$ ,
3.  $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ ,
4.  $\forall j \in \mathbb{Z}, \forall f \in L^2(\mathbb{R}) : f \in V_j \Leftrightarrow D_{2^j} f \in V_0$ ,
5.  $f \in V_0 \Rightarrow \forall m \in \mathbb{Z} : T_m f \in V_0$ ,
6. There exists  $\varphi \in V_0$  such that  $(T_m \varphi)_{m \in \mathbb{Z}}$  is an orthonormal basis of  $V_0$ .

In "6"  $\varphi$  is called scaling function of  $(V_j)_{j \in \mathbb{Z}}$ .

The properties "1"- "6" are somewhat redundant. They are just listed for a better understanding of the construction. The following discussion will successively strip down "1"- "6" to the essential.

**Remarks 8.** (a) Properties "4" and "6" imply that the scaling function  $\varphi$  uniquely determines the multiresolution analysis. We have

$$V_0 = \overline{\text{span}(T_k \varphi : k \in \mathbb{Z})}$$

by "6" (which incidentally takes care of "5") and  $V_j = D_{2^{-j}} V_0$

is prescribed by "4". What are missing are criteria for "1"- "3" and "6". However, note that in the following the focus of attention shifts from the spaces  $V_j$  to the scaling function  $\varphi$ .

(b) The parameter  $j$  can be interpreted as resolution or (inverse) scale or (with some freedom) frequency parameter. Thus the inclusion property "1" has a quite natural interpretation: Increasing resolution amounts to adding information.

If we denote by  $P_j$  the projection on to  $V_j$ , we obtain the characterizations

- "2"  $\Leftrightarrow \forall f \in L^2(\mathbb{R}) : \|f - P_j f\| \rightarrow 0, \text{ as } j \rightarrow \infty$   
 "3"  $\Leftrightarrow \forall f \in L^2(\mathbb{R}) : \|P_j f\| \rightarrow 0, \text{ as } j \rightarrow -\infty$

$P_j f$  can be interpreted as an approximation to  $f$  with resolution  $2^j$ . Thus "2" implies that this approximation converges to  $f$ , as resolution increases.

**Definition 9.** Let  $(V_j)_{j \in \mathbb{Z}}$  denote an MRA. A space  $V_j$  is called approximation space of scale  $j$ . Denote by  $W_j$  the orthogonal complement of  $V_j$  in  $V_{j+1}$ , so that we have the orthogonal decomposition  $W_j \oplus V_j = V_{j+1}$ .  $W_j$  is called the detail space of scale  $j$ .

### Results and discussion

Throughout this section we fix a multiresolution analysis  $(V_j)_{j \in \mathbb{Z}}$  with scaling function  $\varphi$ , wavelet  $\psi$  and scaling coefficients  $(a_k)_{k \in \mathbb{Z}}$ . Given  $f \in V_0$ , we can expand it with respect to two different orthonormal basis:

$$f = \sum_{k \in \mathbb{Z}} c_{0,k} T_k \varphi = \sum_{k \in \mathbb{Z}} d_{-1,k} D_{2^{-1}} T_k \psi + \sum_{k \in \mathbb{Z}} c_{-1,k} D_{2^{-1}} T_k \varphi$$

Here we use the notations

$$c_{j,k} = \langle f, D_{2^{-j}} T_k \varphi \rangle \text{ "approximation coefficient"}$$

$$d_{j,k} = \langle f, D_{2^{-j}} T_k \psi \rangle, \text{ "detail coefficient"}$$

In the following we will use  $d^j = (d_{-j,k})_{k \in \mathbb{Z}}$  and  $c^j = (c_{-j,k})_{k \in \mathbb{Z}}$ . At the heart of the fast wavelet transform are explicit formulae for the correspondence

$$c^0 \leftrightarrow (c^1, d^1)$$

For this purpose we require one more piece of notation.

**Definition 10.** The up and down sampling operators  $\uparrow 2$  and  $\downarrow 2$  are given by

$$(\downarrow 2f)(n) = f(2n), \text{ "down"}$$

$$(\uparrow 2f)(n) = \begin{cases} 0 & n \text{ odd,} \\ f(k) & n = 2k. \end{cases} \text{ "up"}$$

We have  $(\uparrow 2)^* = (\downarrow 2)$ , as well as  $\downarrow 2 \circ \uparrow 2 = I$ , where  $I$  is the identity operator. In particular,  $\uparrow 2$  is an isometry.

**Theorem 11.** The bijection  $c^0 \leftrightarrow (c^1, d^1)$  is given by

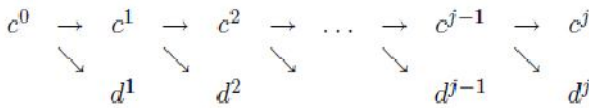
$$\text{Analysis: } c^{j+1} = \downarrow 2(c^j * \ell), d^{j+1} = \downarrow 2(c^j * h)$$

And

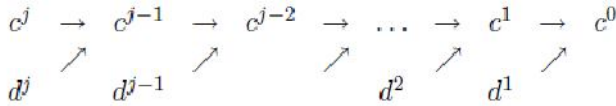
$$\text{Synthesis: } c^j = (\uparrow 2c^{j+1}) * h^* + (\uparrow 2d^{j+1}) * \ell^*$$

Here the filters  $h, \ell$  are given by  $h(k) = (-1)^{1+k} a_{1+k} / \sqrt{2}$  and  $\ell(k) = \overline{a_{-k}} / \sqrt{2}$ .

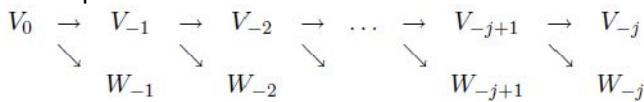
**Remark 12.** The "Cascade Algorithm": Iteration of the analysis step yields  $c^j, d^j$  for  $j \geq 1$ .



Whereas the synthesis step is given by



The decomposition step corresponds to the orthogonal decompositions



For example we give the Haar wavelet as follows:

Example. The Haar wavelet is the following simple step function:

$$\psi(x) = \chi_{[0,1/2[} - \chi_{[1/2,1[} = \begin{cases} 1 & 0 \leq x < 1/2, \\ -1 & 1/2 \leq x < 1 \\ 0 & \text{o. w.} \end{cases}$$

Define

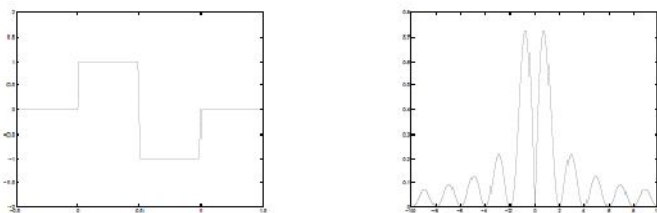
$$V_0 = \{f \in L^2(\mathbb{R}) : \forall n \in \mathbb{Z} : f|_{[n,n+1]} \text{ is constant}\}$$

If we set  $\varphi = \chi_{[0,1]}$ , it is not hard to see that  $(T_k \varphi)_{k \in \mathbb{Z}}$  is an orthonormal basis of  $V_0$ .  $\varphi$  induces an MRA. Note that in this case

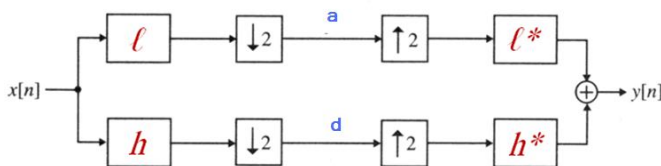
$$V_j = \{f \in L^2(\mathbb{R}) : \forall n \in \mathbb{Z} : f|_{[n/2^j, (n+1)/2^j]} \text{ is constant}\}$$

Thus, the closed subspace  $(V_j)_{j \in \mathbb{Z}}$  is indeed a multiresolution analysis, so called Haar multiresolution analysis. The Haar wavelet and its Fourier are shown in Fig. 1.

Fig. 1. The Haar wavelet and its Fourier transform (only the absolute value).



Following we give a figure of analysis and synthesis of Haar wavelet with convolution.

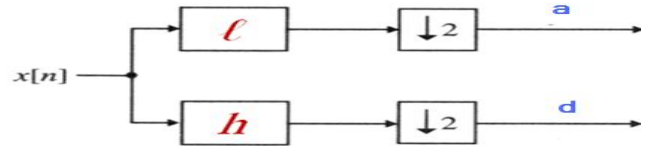


Analysis

We can compute the high pass and low pass filters  $h, \ell$  as follows:

$$\ell = \left[ \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right], \quad h = \left[ -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right]$$

by the following figure

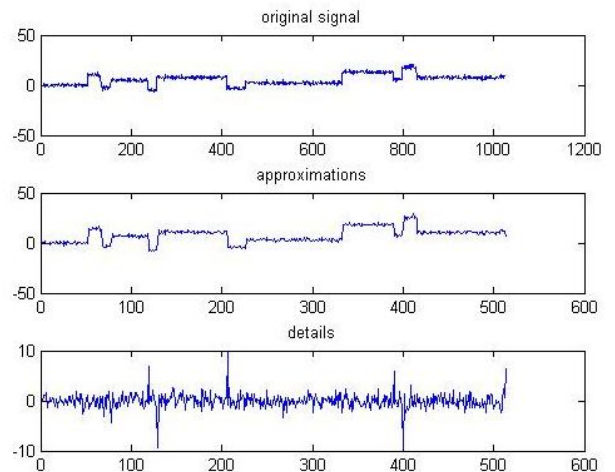


The algorithm in the MATLAB is as follows:

```

1  clc; clear;
2
3  load noisbloc
4  x = noisbloc;
5
6  L = [1/sqrt(2) 1/sqrt(2)];
7  H = [-1/sqrt(2) 1/sqrt(2)];
8
9  A = conv(x,L);
10 D = conv(x,H);
11
12 A = A(1:2:end);
13 D = D(1:2:end);
14
15 figure(1);
16 subplot(3,1,1); plot(x); title('original signal');
17 subplot(3,1,2); plot(A); title('approximation');
18 subplot(3,1,3); plot(D); title('detail');
19 save data.mat A D
    
```

Plots of the analysis of Haar wavelet is as follows:

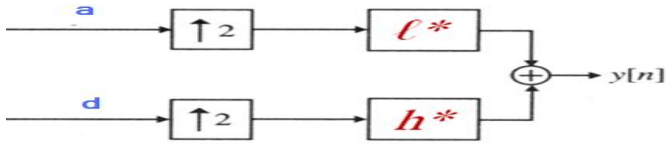


Synthesis

We can compute the high and low pass filters  $h, \ell$  as follows:

$$\ell^* = \left[ \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right], \quad h^* = \left[ \frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \right]$$

by the following figure



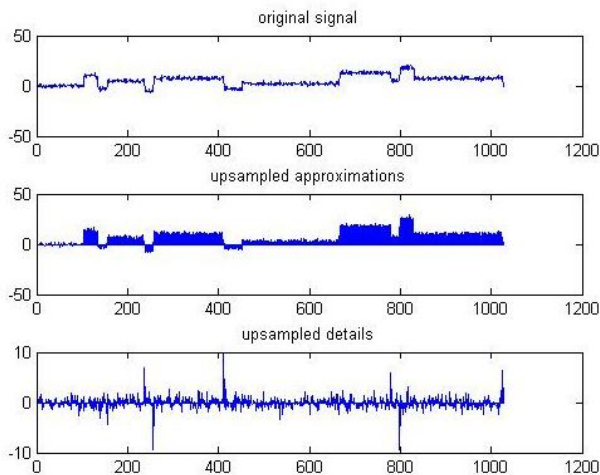
The algorithm in the MATLAB is as follows:

```

1  clc; clear;
2
3  load data.mat
4
5  LR = [1/sqrt(2) 1/sqrt(2)];
6  HR = [1/sqrt(2) -1/sqrt(2)];
7
8  AU = zeros(1, length(A)*2);
9  DU = zeros(1, length(D)*2);
10
11 AU(1:2:end) = A;
12 DU(1:2:end) = D;
13
14 xhat = conv(AU,LR) + conv(DU,HR);
15
16 figure(1);
17 subplot(3,1,1); plot(xhat); title('original signal');
18 subplot(3,1,2); plot(AU); title('upsampled approximation');
19 subplot(3,1,3); plot(DU); title('upsampled detail');
20

```

Plots of the synthesis of Haar wavelet is as follows:



## Conclusion

In this study the analysis of Haar's wavelet is shown by MATLAB programming. The base of this study is followed from multiresolution analysis. Approximation and detail of synthesis of this wavelet is obtained by the at least error via multiresolution analysis.

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